b) Determine if c for Figure 4.2(c) is positive or negative. What about for Figure 4.2(d)?

7. This exercise involves the proof of some of the statements made about the integral curves when the eigenvalues are complex-valued. The matrix $\mathbf{A}$ is given in (4.4), and it is assumed that $\mu \neq 0$.

a) Assuming that d_1 and d_2 are not both zero, from (4.27) and the identity $\cos^2(\mu t) + \sin^2(\mu t) = 1$, show that

$$(p_2^2 + q_2^2)x^2 - 2p_1p_2xy + p_1^2y^2 = k^2e^{2\lambda t},$$

where $p_1 = b$, $p_2 = \lambda - a$, $q_2 = \mu$, and k is a positive constant.

b) In part (a), what equation is obtained when there are imaginary eigenvalues? Explain why this verifies the statement that when there are imaginary eigenvalues the solution curve is an ellipse centered at the origin.

c) Show that to have a circular curve requires $b \neq 0$ and $c = -b$.

d) Explain why you get a spiral in the case of when $\lambda \neq 0$.

4.7 ▪ Stability

The phase plane can be useful for visualizing stability or instability of a steady state solution. To explain how, recall from Section 2.4 that a **steady state** is a constant that satisfies the differential equation. So, for the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, a steady state is a constant vector $\mathbf{x}_s$ that satisfies $\mathbf{A}\mathbf{x}_s = \mathbf{0}$. A solution of this equation is $\mathbf{x}_s = \mathbf{0}$. This is the only solution if $\mathbf{A}$ is invertible. Given the prominent role eigenvalues play in this chapter, it is useful to know that $\mathbf{A}$ is invertible if, and only if, $r = 0$ is not an eigenvalue for $\mathbf{A}$.

The definitions of unstable and asymptotically stable are effectively the same as in Section 2.4. Namely, a steady state $\mathbf{x}_s$ is **asymptotically stable** if any initial value $\mathbf{x}(0)$ chosen near $\mathbf{x}_s$ results in

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_s. \tag{4.39}$$

The steady state is **unstable** if, no matter how close to $\mathbf{x}_s$ you restrict the choice for $\mathbf{x}(0)$, it is always possible to find an initial value $\mathbf{x}(0)$ that results in the solution $\mathbf{x}(t)$ becoming unbounded as t increases.

It is easy to determine stability using the phase plane. For example, in Table 4.1(a), when $r_1 > 0$ and $r_2 > 0$, the arrows on the integral curves indicate movement out away from the origin. Consequently, this is an example of when $\mathbf{x}_s = \mathbf{0}$ is unstable. Conversely, when $r_1 < 0$ and

$r_2 < 0$, the flow in towards the origin, and this means $\mathbf{x}_s = \mathbf{0}$ is asymptotically stable. In fact, looking at the various possibilities in Table 4.1, you conclude that if $\mathbf{A}$ has an eigenvalue with $\text{Re}(r) > 0$, then the steady state in unstable. Similarly, if the eigenvalues of $\mathbf{A}$ are both negative, or if $\text{Re}(r) < 0$, then the steady state is asymptotically stable.

Our classification of a steady state being unstable or asymptotically stable does not include what happens when the eigenvalues are imaginary. As shown in Table 4.1(d), the solution does not decay to zero, or blowup, but simply encircles the origin. So, the steady state is stable but it is not asymptotically stable. In this case, $\mathbf{x}_s$ is said to be **neutrally stable**.

The conclusions in the previous paragraphs were made using the phase portraits in Table 4.1. A more rigorous derivation is possible using the formulas for the general solution given in (4.26) and (4.28). Whichever approach you take, the resulting conclusions are summarized in the following theorem.

Stability Theorem for a Linear System. *For $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 matrix, the following hold:*

1. *If all of the eigenvalues of $\mathbf{A}$ satisfy $Re(r) < 0$, then $\mathbf{x}_s = \mathbf{0}$ is an asymptotically stable steady state.*
2. *If $\mathbf{A}$ has one or more eigenvalues with $Re(r) > 0$, then $\mathbf{x}_s = \mathbf{0}$ is an unstable steady state.*
3. *If the eigenvalues of $\mathbf{A}$ are imaginary, so $r = \pm i\mu$ with $\mu \neq 0$, then $\mathbf{x}_s = \mathbf{0}$ is a neutrally stable steady state.*

The first two conclusions in the above theorem hold when $\mathbf{A}$ is $n \times n$. The third, however, must be modified. One version is: If $\mathbf{A}$ is nondefective, has two or more imaginary eigenvalues, and all eigenvalues satisfy $\text{Re}(r) \leq 0$, then $\mathbf{x}_s = \mathbf{0}$ is a neutrally stable steady state.

In addition to their stability, steady states are often identified by the geometric properties of the solution near the steady state. So, for example, because of the outward direction of the flow in Figure 4.1(a), the steady state is called a **source**. In contrast, because of the inward flow in Figure 4.1(c), the steady state is called a **sink**. For similar reasons, the flow in Figure 4.1(e) is a **spiral source**, and the one in Figure 4.1(f) is a **spiral sink**. Finally, the steady state in Figure 4.1(b) is a **saddle**, and the one in Figure 4.1(d) is a **center**.

Example 1: Determine the stability of the steady state $\mathbf{x}_s = \mathbf{0}$ for

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} \mathbf{x}.$$

Answer: The characteristic equation for the matrix is $r^2+3r-10=0$, and from this it follows that the eigenvalues are $r=-5$ and $r=2$. Given that there is at least one eigenvalue that is positive, $\mathbf{x}_s=\mathbf{0}$ is unstable. Moreover, since it has one positive, and one negative, eigenvalue, the steady state is a saddle point. ■

Example 2: Determine the stability of the steady state $\mathbf{x}_s=\mathbf{0}$ for

$$\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix}\mathbf{x}.$$

Answer: The characteristic equation for the matrix is $r^2+r+4=0$, and from this it follows that the eigenvalues are $r=\frac{1}{2}(-1\pm i\sqrt{15})$. Given that both have negative real part, then $\mathbf{x}_s=\mathbf{0}$ is asymptotically stable. Moreover, since the eigenvalues are complex with negative real part, the steady state is a spiral sink. ■

Example 3: Find the steady state, and determine its stability for

$$\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \tag{4.40}$$

Steady State: Since a steady state is a constant vector that satisfies the differential equation, we require that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In component form,

$$\begin{aligned} u_1 + u_2 &= -2, \\ u_1 - u_2 &= -4. \end{aligned}$$

From the second equation, $u_1=-4+u_2$ and, after substituting this into the first equation, one finds that $u_2=1$. So, $u_1=-3$, and this gives the steady state

$$\mathbf{u}_s = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Stability: Letting $\mathbf{u}=\mathbf{u}_s+\mathbf{x}$, and substituting this into the differential equation, one finds that $\mathbf{x}'=\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is the matrix in (4.40). If $\mathbf{x}_s=\mathbf{0}$ is unstable, then so is $\mathbf{u}_s$. Similarly, if $\mathbf{x}_s=\mathbf{0}$ is asymptotically stable, then $\mathbf{u}_s$ asymptotically stable. Now, the characteristic equation for $\mathbf{A}$ is $r^2-2=0$. From this, the eigenvalues are found to be $r=\pm\sqrt{2}$. Given that one is positive, $\mathbf{x}_s$ is unstable, and therefore $\mathbf{u}_s$ is unstable. Moreover, since it has one positive, and one negative, eigenvalue, $\mathbf{u}_s$ is a saddle point. ■

Exercises

1. Determine whether $\mathbf{x}_s = \mathbf{0}$ is an asymptotically stable, unstable, or neutrally stable steady state for the following differential equations. Also, state whether the steady state is a sink, source, spiral sink, spiral source, saddle, or center.

   a) $\mathbf{x}' = \begin{pmatrix} -1 & 6 \\ 1 & 0 \end{pmatrix}\mathbf{x}$ d) $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\mathbf{x}$ g) $\mathbf{x}' = \begin{pmatrix} 2 & 5 \\ -5 & -6 \end{pmatrix}\mathbf{x}$

   b) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\mathbf{x}$ e) $\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 6 & -6 \end{pmatrix}\mathbf{x}$ h) $\mathbf{x}' = \begin{pmatrix} 0 & -9 \\ 1 & 0 \end{pmatrix}\mathbf{x}$

   c) $\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$ f) $\mathbf{x}' = \begin{pmatrix} 1 & \frac{1}{4} \\ -5 & 0 \end{pmatrix}\mathbf{x}$ i) $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

2. Write the following as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and then determine whether $\mathbf{x}_s = \mathbf{0}$ is an asymptotically stable, unstable, or neutrally stable steady state.

   a) The simple harmonic oscillator given in (3.47).
   b) The damped oscillator given in (3.54), with $c > 0$.
   c) It is usually stated that negative damping is unstable. For the mass-spring-dashpot system, negative damping means that c is negative. Is the system unstable in this case?

3. Find the steady state $\mathbf{u}_s$, and determine its stability, for the following differential equations. Also, state whether the steady state is a sink, source, spiral sink, spiral source, saddle, or center.

   a) $\mathbf{u}' = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ c) $\mathbf{u}' = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{u} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   b) $\mathbf{u}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ d) $\mathbf{u}' = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. The blue x's in the plots in Figure 4.5 show possible locations, in the complex plane, of the two eigenvalues of a matrix $\mathbf{A}$. For example, plot (a) indicates the eigenvalues are real-valued, one positive and one negative. For plots (a)-(d), determine whether $\mathbf{x}_s = \mathbf{0}$ is an asymptotically stable, unstable, or neutrally stable steady state, and state whether the steady state is a sink, source, spiral sink, spiral source, saddle, or center. For plots (e) and (f) explain why the eigenvalues can not be in the locations that are shown.

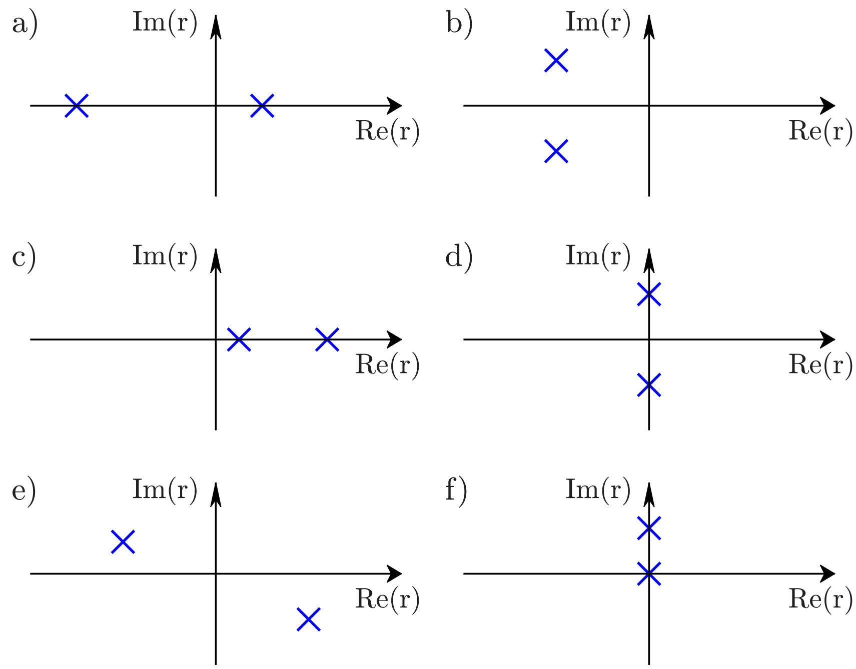

Figure 4.5. *The x's show possible locations of the eigenvalues of the matrix* $\mathbf{A}$ *in the complex plane, as explained in Exercise 4.*

5. This exercise contains useful information to determine the stability of $\mathbf{x}_s = \mathbf{0}$ without having to calculate eigenvalues. Assume that $\mathbf{A}$ is given in (4.4) and that $\det(\mathbf{A}) \neq 0$. Also, the *trace of a matrix* is the sum of the numbers on the diagonal. The formula is $\mathrm{tr}(\mathbf{A}) = a + d$.
   a) Show that the eigenvalues of $\mathbf{A}$ are $\frac{1}{2}\left[\mathrm{tr}(\mathbf{A}) \pm \sqrt{[\mathrm{tr}(\mathbf{A})]^2 - 4\det(\mathbf{A})}\right]$.
   b) Explain why $r = 0$ is not an eigenvalue for $\mathbf{A}$.
   c) Show that if $\mathrm{tr}(\mathbf{A}) > 0$, then $\mathbf{x}_s$ is unstable.
   d) Show that if $\det(\mathbf{A}) < 0$, then $\mathbf{x}_s$ is unstable.
   e) Show that if $\mathrm{tr}(\mathbf{A}) = 0$, and $\det(\mathbf{A}) > 0$, then $\mathbf{x}_s$ is neutrally stable.
   f) Show that if $\mathrm{tr}(\mathbf{A}) < 0$ and $\det(\mathbf{A}) > 0$, then $\mathbf{x}_s$ is asymptotically stable.

6. Determine whether $\mathbf{x}_s = \mathbf{0}$ is an asymptotically stable, unstable, or neutrally stable steady state for the following differential equations.

$$\text{a)}\ \mathbf{x}' = \begin{pmatrix} -1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}\mathbf{x} \qquad \text{b)}\ \mathbf{x}' = \begin{pmatrix} -3 & 0 & 0 \\ 1 & -1 & 5 \\ 4 & 0 & -2 \end{pmatrix}\mathbf{x}$$

c) $\mathbf{x}' = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x}$

e) $\mathbf{x}' = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -5 & -1 \end{pmatrix} \mathbf{x}$

d) $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 0 & -1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \mathbf{x}$

f) $\mathbf{x}' = \begin{pmatrix} -5 & 0 & 2 \\ 0 & -2 & 0 \\ -5 & 0 & 1 \end{pmatrix} \mathbf{x}$

4.8 ▪ Modeling

Many of the modeling problems we have considered in the earlier chapters can be generalized so they result in a system of differential equations. This includes the mixing model and the Newton's Second Law material discussed in Section 2.3, and the oscillator models considered in Section 3.10. It is recommended that you review these earlier examples.

In some of the modeling problems to follow, it is required to solve the inhomogeneous differential equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}, \tag{4.41}$$

where $\mathbf{b}$ is a constant vector. Because this is a linear equation, the general solution can be written as $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is the general solution of the associated homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and $\mathbf{x}_p$ is a particular solution of 4.41. Finding $\mathbf{x}_h$ was considered earlier and a summary is given in Section 4.5.

A particular solution can be obtained using undetermined coefficients (see Section 3.7). Because $\mathbf{b}$ is a constant vector, the assumption is that the particular solution $\mathbf{x}_p$ is a constant. This means that $\mathbf{x}_p' = \mathbf{0}$, and inserting this into 4.41 we get that

$$\mathbf{A}\mathbf{x}_p = -\mathbf{b}. \tag{4.42}$$

Any solution of this equation can be used as a particular solution.

Be aware that it is possible that (4.42) does not have a solution. For this to happen, $r = 0$ must be an eigenvalue of $\mathbf{A}$. To explain how the assumed form of the particular solution must be modified, assume $\mathbf{A}$ has eigenvalues $r_1 = 0$ and $r_2 \neq 0$, with respective eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Using the eigenvectors as a basis, we can write

$$\mathbf{x} = \alpha_1(t)\mathbf{a}_1 + \alpha_2(t)\mathbf{a}_2. \tag{4.43}$$

Similarly, we can find constants β_1 and β_2 so that $\mathbf{b} = \beta_1\mathbf{a}_1 + \beta_2\mathbf{a}_2$. Substituting these eigenvector expressions into (4.41), and using the fact that $\mathbf{A}\mathbf{a}_j = r_j\mathbf{a}_j$, we obtain $\alpha_1'\mathbf{a}_1 + \alpha_2'\mathbf{a}_2 = \beta_1\mathbf{a}_1 + (\beta_2 + \alpha_2 r_2)\mathbf{a}_2$. This can be rearranged to give

$$(\alpha_1' - \beta_1)\mathbf{a}_1 + (\alpha_2' - \beta_2 - \alpha_2 r_2)\mathbf{a}_2 = \mathbf{0}.$$